

The Cauchy problem for the generalized hyperbolic Novikov-Veselov equation

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We begin by introducing a new procedure for construction of the exact solutions to Cauchy problem of the real-valued (hyperbolic) Novikov-Veselov equation. The procedure shown therein utilizes the well-known Airy function $\text{Ai}(\xi)$ which in turn serves as a solution to the ordinary differential equation $\frac{d^2 z}{d\xi^2} = \xi z$. In the second part of the article we show that the aforementioned procedure can also work for the n -th order generalizations of the Novikov-Veselov equation, provided that one replaces the Airy function with the appropriate solution of the ordinary differential equation $\frac{d^{n-1} z}{d\xi^{n-1}} = \xi z$.

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I. INTRODUCTION

The Novikov-Veselov equation (NV) first came into being almost 30 years ago, in 1984, when Sergey Novikov and Aleksander Veselov have introduced it as a two-dimensional version of the Korteweg-de Vries equation [NV]. Since then, a huge and ever-growing body of works related to the study of NV equations has been established (see, for example, [GN], [LMSS], [P] and also [CMMPS] for a rather extensive review of a recent literature on the subject). One of the most prominent aspects of this interest was an almost universal adoption of an inverse scattering method as a primary tool for conducting the research and finding the exact solutions of NV. However, in this article we wish to discuss an alternative method of solving the Cauchy problem for NV. This method, albeit simple in principle, appears to be deep enough to be applicable to a rather broad class of equations, NV and the 2-dimensional nonlinear Schrödinger equation being its two prominent members.

II. THE MOUTARD TRANSFORMATION

Let us start by introducing the hyperbolic NV equation:

$$\begin{aligned} u_t &= u_{xxx} + u_{yyy} + 3((au)_x + (bu)_y) \\ u_x &= a_y, \quad u_y = b_x. \end{aligned} \tag{1}$$

This system allows for a Lax pair of the following type:

$$\begin{aligned} \Psi_{xy} + u\Psi &= 0 \\ \Psi_t &= \Psi_{xxx} + \Psi_{yyy} + 3(a\Psi_x + b\Psi_y). \end{aligned} \tag{2}$$

If one knows two linearly independent solutions $\Psi_1(x, y, t)$ and $\Psi_2(x, y, t)$ for (2), then one can utilize the famous Moutard transformation to construct a new function $\Psi[1](x, y, t)$ that will serve as a solution to the same equation (2) albeit with a new potential $u[1](x, y, t)$. The new potential will then satisfy the relation

$$u[1] = u + 2\partial_x \partial_y \ln \Psi_1. \tag{3}$$

Let us assume that $u = a = b = 0$. Then the entire system (2) simplifies to

$$\Psi_{xy} = 0 \tag{4}$$

$$\Psi_t = \Psi_{xxx} + \Psi_{yyy}. \tag{5}$$

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The equation (4) can be resolved by separating the variables. The resulting solution will be of a form:

$$\Psi_1(x, y, t) = A(x, t) + B(y, t), \quad (6)$$

where A, B are two arbitrary functions that are continuously differentiable by x and y , correspondingly. Substituting (6) into (3) yields a following post-Moutard form of function $u[1](x, y, t)$:

$$u[1] = -2 \frac{\partial_x A \cdot \partial_y B}{(A + B)^2}. \quad (7)$$

As follows from (7), our next goal should lie in ascertaining the exact forms of the functions $A(x, t)$ and $B(y, t)$. This task can be accomplished by looking at the equation (5) which we have ignored so far. We will rewrite it as a standard Cauchy problem by introducing the initial conditions for $A(t, x), B(t, y)$

$$A(0, x) = \phi(x), \quad B(0, y) = \Phi(y). \quad (8)$$

and rewriting the (5) as a system

$$\begin{aligned} A_t &= A_{xxx} + T(t) \\ B_t &= B_{yyy} - T(t), \end{aligned} \quad (9)$$

where $T = T(t)$ is an arbitrary time-dependent function. The apparently symmetric nature of (9) allows us to restrict our attention on just one of the equations therein, namely – the first one.

We begin by introducing the Fourier transform $\tilde{A}(p, t)$ of the function $A(x, t)$:

$$\tilde{A}(p, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(x, t) e^{-ipx} dx.$$

This transformation is handy because of the identity

$$A(p, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{A}(p, t) e^{ipx} dp, \quad (10)$$

which, after being substituted into (9), yields the equation

$$\int_{-\infty}^{\infty} \left(\frac{\partial \tilde{A}}{\partial t} + ip^3 \tilde{A} - T \right) e^{ipx} dp = 0. \quad (11)$$

The equation (11) must be satisfied for all x and p , and therefore leads to:

$$\frac{\partial \tilde{A}}{\partial t} + ip^3 \tilde{A} = T(t). \quad (12)$$

(12) is a nonhomogeneous linear O.D.E. of first order. Its general solution is

$$\tilde{A}(p, t) = C(p) e^{-ip^3 t} + \int_0^t T(\tau) e^{-ip^3(t-\tau)} d\tau, \quad (13)$$

where $C(p)$ is a function, determinable from the initial conditions (8). Using the inverse Fourier transform (10) we come to the following conclusion:

$$A(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int_0^t T(\tau) e^{ip^3 \tau} d\tau + C(p) \right) e^{ipx - ip^3 t} dp. \quad (14)$$

According to (8),

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} C(p) e^{ipx} dp,$$

so the unknown $C(p)$ is an inverse Fourier transform of the initial condition $\phi(x)$, i.e.:

$$C(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x) e^{-ipx} dx.$$

and we subsequently end up with the following general formula for the function $A(x, t)$:

$$A(x, t) = \frac{1}{\sqrt{2\pi}} \int_0^t T(\tau) d\tau \int_{-\infty}^{\infty} e^{ipx - ip^3(t-\tau)} dp + \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(\xi) d\xi \int_{-\infty}^{\infty} e^{ip(x-\xi) - ip^3 t} dp. \quad (15)$$

The (15) can be further simplified by pointing out the similarity between the integrals with respect to variable p and the *Airy function* $\text{Ai}(\xi)$. The Airy function is a particular solution of the eponymous Airy equation:

$$\frac{d^2 z}{d\xi^2} = \xi z, \quad (16)$$

that has a following integral representation:

$$\text{Ai}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\left(\frac{t^3}{3} + \xi t\right)} dt. \quad (17)$$

Using this fact together with the apparent identity:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipa - ip^3 b} dp = \frac{1}{\sqrt[3]{3b}} \text{Ai}\left(\frac{a}{\sqrt[3]{3b}}\right),$$

together with the equation (15) and the similar one written for $B(y, t)$ finally yields:

$$\begin{aligned} A(x, t) &= \int_0^t \frac{T(\tau)}{\sqrt[3]{3(\tau-t)}} \text{Ai}\left(\frac{x}{\sqrt[3]{3(\tau-t)}}\right) d\tau + \frac{1}{\sqrt{2\pi}\sqrt[3]{3t}} \int_{-\infty}^{\infty} \phi(\xi) \text{Ai}\left(\frac{\xi-x}{\sqrt[3]{3t}}\right) d\xi \\ B(y, t) &= - \int_0^t \frac{T(\tau)}{\sqrt[3]{3(\tau-t)}} \text{Ai}\left(\frac{y}{\sqrt[3]{3(\tau-t)}}\right) d\tau + \frac{1}{\sqrt{2\pi}\sqrt[3]{3t}} \int_{-\infty}^{\infty} \Phi(\eta) \text{Ai}\left(\frac{\eta-y}{\sqrt[3]{3t}}\right) d\eta. \end{aligned} \quad (18)$$

So, we end up with both the solution $\Psi_1 = A + B$ of the Lax pair (4), (5), and, as a courtesy of Moutard transform (3), with a solution $u[1]$ of the NV equation (1) as well. In other words, to find a non-zero solution of the NV equation, it will suffice to start with $u \equiv 0$, impose the boundary conditions (8) on the Lax pair (4), (5), use (18) to find its solution and conclude the calculations by finding a function $u[1]$ via the Moutard transformation (3). As straightforward as it is, there is one question we should ask: what would happen should we try to *invert* the process and instead start out with the boundary conditions for the NV equation itself?

III. THE CAUCHY PROBLEM FOR THE NOVIKOV-VESELOV EQUATION

In the previous chapter we have shown that there shall exist a solution $u[1](x, y, t)$ to the NV equation, whose exact form can be derived via the Moutard transformation (7) from the solutions of the system (4, 5), provided we are given the initial conditions (8). But what would happen if the exact forms of the functions $\phi(x)$ and $\Phi(y)$ are *unknown* and we are instead given the initial conditions for the NV equation itself, and would it still be possible to find the required $u[1]$? In other words, is it possible to find an analytic solution to the Cauchy problem for the NV equation provided we only know that the solution has a general structure (7)? In short, the answer is “yes”.

Let us start by introducing the set of conditions and boundary conditions for the NV equation:

$$\begin{aligned} u[1](x, y, 0) &= u_0(x, y) \\ u_0(x, 0) &= A_1(x) \\ u_0(0, y) &= B_1(y) \\ A_1(0) &= B_1(0) = C, \end{aligned} \quad (19)$$

where $C \in \mathbb{R}$ is some constant that is given to us together with the boundary conditions A_1 and B_1 . Since we know that $u[1]$ satisfies the Moutard transformation, we also know that:

$$u_0(x, y) = -2 \frac{\phi'(x) \cdot \dot{\Phi}(y)}{(\phi(x) + \Phi(y))^2}, \quad (20)$$

where ϕ and Φ are defined as in Sec II, and $'$ and \cdot denote the partial derivatives with respect to x and y variables correspondingly. From (20) and (19) it immediately follows that

$$\begin{aligned} A_1(x) &= -2 \frac{\phi'(x) \cdot \dot{\Phi}(0)}{(\phi(x) + \Phi(0))^2} \\ B_1(y) &= -2 \frac{\phi'(0) \cdot \dot{\Phi}(y)}{(\phi(0) + \Phi(y))^2} \\ C &= -2 \frac{\phi'(0) \cdot \dot{\Phi}(0)}{(\phi(0) + \Phi(0))^2}. \end{aligned} \quad (21)$$

The first two differential equations in (21) can be easily integrated; for example, the first one after the integration with respect to the variable x yields

$$\frac{\dot{\Phi}(0)}{\phi(x) + \Phi(0)} = \frac{1}{2} \int_0^x A_1(\xi) d\xi + -\frac{\dot{\Phi}(0)}{\phi(0) + \Phi(0)},$$

which leads us to the following conclusion:

$$\begin{aligned} \phi(x) &= \frac{2\dot{\Phi}_0}{\int_0^x A_1(\xi) d\xi + \frac{2\dot{\Phi}_0}{\phi_0 + \Phi_0}} - \Phi_0 \\ \phi'(x) &= \frac{-2\dot{\Phi}_0 A_1(x)}{\left(\int_0^x A_1(\xi) d\xi + \frac{2\dot{\Phi}_0}{\phi_0 + \Phi_0} \right)^2}, \end{aligned} \quad (22)$$

where we have introduced the notation: $\phi(0) = \phi_0$, $\Phi(0) = \Phi_0$, $\phi'(0) = \phi'_0$ and $\dot{\Phi}(0) = \dot{\Phi}_0$. In a similar fashion, the boundary condition $\Phi(y)$ and its derivative will satisfy:

$$\begin{aligned} \Phi(y) &= \frac{2\phi'_0}{\int_0^y B_1(\zeta) d\zeta + \frac{2\phi'_0}{\phi_0 + \Phi_0}} - \phi_0 \\ \dot{\Phi}(y) &= \frac{-2\phi'_0 B_1(y)}{\left(\int_0^y B_1(\zeta) d\zeta + \frac{2\phi'_0}{\phi_0 + \Phi_0} \right)^2}. \end{aligned} \quad (23)$$

The system (22, 23) depends on four constants: ϕ_0 , Φ_0 , ϕ'_0 and $\dot{\Phi}_0$. Three of them can be chosen arbitrarily, whereas the fourth one would have to satisfy the equation (21), namely:

$$-2 \frac{\phi'_0 \cdot \dot{\Phi}_0}{(\phi_0 + \Phi_0)^2} = C.$$

Curiously, this choice does not affect the Cauchy problem of the NV equation in the least, for it can be shown by direct substitution into (20) that:

$$u_0(x, y) = \frac{4C A_1(x) B_1(y)}{\left(\int_0^x \int_0^y A_1(\xi) B_1(\zeta) d\zeta d\xi + 2C \right)^2}, \quad (24)$$

i.e. the *initial* condition $u_0(x, y)$ depends only on the known *initial boundary* conditions $A_1(x)$, $B_1(y)$ and C .

We are now ready to answer the question posed in the beginning of this section: provided we know the initial conditions (24), how do we solve the corresponding Cauchy problem of the hyperbolic real-valued Novikov-Veselov equation? The answer lies in *repeating* the Moutard transformation process we described in Sec. III! Indeed, since the unknown functions $\phi(x)$ and $\Phi(y)$ satisfy the relations (22) and (23), all we really have to do is substitute them into the system (18), derive $A(x, t)$ and $B(y, t)$, and substitute them in equation (7) to find out the sought after $u[1](x, y, t)$, which will conclude the problem.

Lets summarize everything we have said so far. In order to find an exact solution $u(x, y, t)$ to the hyperbolic real-valued Novikov-Veselov equation

$$u_t = u_{xxx} + u_{yyy}, \quad (25)$$

with the given initial boundary conditions

$$u(x, 0, 0) = A_1(x), \quad u(0, y, 0) = B_1(y), \quad u(0, 0, 0) = C,$$

that correspond to the initial condition

$$u_0(x, y) = \frac{4CA_1(x)B_1(y)}{\left(\int_0^x \int_0^y A_1(\xi)B_1(\zeta)d\zeta d\xi + 2C\right)^2},$$

one shall:

1. Choose a differentiable function $T(t)$ and four numbers α, β, γ and δ that satisfy the condition,

$$-2\frac{\gamma \cdot \delta}{(\alpha + \beta)^2} = C.$$

2. Find two support function $\phi(x)$ and $\Phi(y)$ via the formulas

$$\begin{aligned} \phi(x) &= \frac{2\delta}{\int_0^x A_1(\xi)d\xi + \frac{2\delta}{\alpha+\beta}} - \beta \\ \Phi(y) &= \frac{2\gamma}{\int_0^y B_1(\zeta)d\zeta + \frac{2\gamma}{\alpha+\beta}} - \alpha. \end{aligned}$$

3. Substitute $\phi(x)$ and $\Phi(y)$ into the equations

$$\begin{aligned} A(x, t) &= \int_0^t \frac{T(\tau)}{\sqrt[3]{3(\tau-t)}} \text{Ai}\left(\frac{x}{\sqrt[3]{3(\tau-t)}}\right) d\tau + \frac{1}{\sqrt{2\pi}\sqrt[3]{3t}} \int_{-\infty}^{\infty} \phi(\xi) \text{Ai}\left(\frac{\xi-x}{\sqrt[3]{3t}}\right) d\xi \\ B(y, t) &= -\int_0^t \frac{T(\tau)}{\sqrt[3]{3(\tau-t)}} \text{Ai}\left(\frac{y}{\sqrt[3]{3(\tau-t)}}\right) d\tau + \frac{1}{\sqrt{2\pi}\sqrt[3]{3t}} \int_{-\infty}^{\infty} \Phi(\eta) \text{Ai}\left(\frac{\eta-y}{\sqrt[3]{3t}}\right) d\eta. \end{aligned} \quad (26)$$

4. Substitute the new functions $A(x, t)$ and $B(y, t)$ into the equation

$$u = -2\frac{\partial_x A \cdot \partial_y B}{(A + B)^2}. \quad (27)$$

The resulting function $u(x, y, t)$ will be a proper solution of the Cauchy problem since by construction it will satisfy both the NV equation (25), and the initial conditions $u(x, y, 0) = u_0(x, y)$. We would like to emphasize here that this procedure does not involve anything more complicated than partial differentiation and integration and can therefore be used for both the analytic study of the properties of the solutions of NV equation and the corresponding numerical calculations.

IV. GENERALIZATION OF THE METHOD: THE HIGHER ORDER EQUATIONS

Let us now say a few words about the more general problem. Suppose we have the following operator-type Lax pair:

$$\begin{aligned}\partial_x \partial_y \Psi + u \Psi &= 0 \\ \partial_t \Psi &= \partial_x^n \Psi + \partial_y^n \Psi,\end{aligned}\tag{28}$$

where $n \in \mathbb{N}^+$ in some non-zero natural number. This system will correspond to a family of Lax equations, with the special case $n = 3$ corresponding to the hyperbolic NV equation. It will still allow for the Moutard transformation, and therefore the crux of our discussion would still be applicable for arbitrary n . However, one thing that *must* change is the exact form of the equations for $A(x, t)$ and $B(y, t)$. The formula (26) is no longer applicable for the general case and should be properly replaced. In order to find out the suitable replacement, we shall separately consider two alternative cases: when n is odd and when n is even.

Case 1: Odd n . Let $n = 2m + 1$, where $m \geq 0$. Following our previous discussion, let us consider the special case $u \equiv 0$. Then the system (28) turns into

$$\begin{aligned}\partial_x \partial_y \Psi &= 0 \\ \partial_t \Psi &= \partial_x^n \Psi + \partial_y^n \Psi.\end{aligned}$$

Since the first equation requires that $\Psi = A(x, t) + B(y, t)$, the system subsequently splits into the following equations:

$$\partial_t A_{2m+1} = \partial_x^{2m+1} A_{2m+1}, \quad \partial_t B_{2m+1} = \partial_y^{2m+1} B_{2m+1},$$

where for simplicity we have omitted the arbitrary function $T(t)$. Using the Fourier transformation

$$\tilde{A}_{2m+1}(p, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A_{2m+1}(x, t) e^{-ipx} dx,$$

we end up with the differential equation

$$\partial_t \tilde{A}_{2m+1} = (ip)^{2m+1} \tilde{A}_{2m+1} = i(-1)^m p^{2m+1} \tilde{A}_{2m+1}.$$

Solving it and returning back to $A(x, t)$ as described in Sec. II yields

$$A_{2m+1}(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\xi \phi(\xi) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp e^{i(p(x-\xi) + (-1)^m p^{2m+1} t)}.\tag{29}$$

As we know, in the special case $m = 1$ (i.e. $n = 3$) the inner integral in (32) can be rewritten in terms of the Airy function

$$\text{Ai}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\left(\frac{t^3}{3} + \xi t\right)} dt = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \cos\left(\xi t + \frac{t^3}{3}\right) dt,$$

which serves as a solution to the Airy equation

$$\frac{d^2 z}{d\xi^2} = \xi z,$$

and is easily derived using either Fourier or Laplace transformation [6].

Similarly, it is easy to show that one of a solutions to a more general equation

$$\frac{d^{2m} z}{d\xi^{2m}} = \xi z,$$

will be a *higher-order generalization* of the Airy function [7]:

$$\text{Ai}_{2m+1}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\left(\xi t - \frac{(-1)^m}{2m+1} t^{2m+1}\right)} dt = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \cos\left(\xi t - \frac{(-1)^m}{2m+1} t^{2m+1}\right) dt, \quad (30)$$

which means that the required functions A_{2m+1} and B_{2m+1} can be derived from the initial conditions $\phi(x)$ and $\Phi(y)$ by the following formulas:

$$\begin{aligned} A_{2m+1}(x, t) &= \frac{1}{2^{m+1}\sqrt{(2m+1)t}} \int_{-\infty}^{\infty} d\xi \phi(\xi) \text{Ai}_{2m+1}\left(\frac{\xi - x}{2^{m+1}\sqrt{(2m+1)t}}\right), \\ B_{2m+1}(x, t) &= \frac{1}{2^{m+1}\sqrt{(2m+1)t}} \int_{-\infty}^{\infty} d\zeta \Phi(\zeta) \text{Ai}_{2m+1}\left(\frac{\zeta - y}{2^{m+1}\sqrt{(2m+1)t}}\right). \end{aligned} \quad (31)$$

Case 2: Even n . Let $n = 2m$, where $m \geq 0$. This time let us utilize not a Fourier but a Laplace transform:

$$A_{2m}(x, t) = \int_{-\infty}^{\infty} \tilde{A}_{2m}(p, t) e^{px} dx.$$

the equation for $\tilde{A}(p, t)$ is

$$\frac{\partial \tilde{A}_{2m}}{\partial t} = p^{2m} \tilde{A}_{2m},$$

so the required function $A(x, t)$ will satisfy the equation

$$A_{2m}(x, t) = \int_{-\infty}^{\infty} d\xi \phi(\xi) \int_{-\infty}^{\infty} dp e^{p(x-\xi)+p^{2m}t}. \quad (32)$$

It is not difficult to show that the Laplace transformation method applied to the ordinary differential equation

$$\frac{d^{2m-1}z}{d\xi^{2m-1}} = \xi z,$$

will yield a following solution

$$\text{Ai}_{2m}(\xi) = \int_{-\infty}^{\infty} \exp\left(\xi t - \frac{t^{2m}}{2m}\right) dt, \quad (33)$$

and so the even case produces the formulas that are quite similar to the old ones, namely:

$$\begin{aligned} A_{2m}(x, t) &= \frac{1}{2^m\sqrt{-2mt}} \int_{-\infty}^{\infty} d\xi \phi(\xi) \text{Ai}_{2m}\left(\frac{x - \xi}{2^m\sqrt{-2mt}}\right), \\ B_{2m}(x, t) &= \frac{1}{2^m\sqrt{-2mt}} \int_{-\infty}^{\infty} d\zeta \Phi(\zeta) \text{Ai}_{2m}\left(\frac{y - \zeta}{2^m\sqrt{-2mt}}\right). \end{aligned} \quad (34)$$

Note the appearance of a negative sign under the root in (34), which serves as a indication of an ill-posedness of our problem for $t > 0$.

BIBLIOGRAPHY

- [NV] Novikov, S.P. and Veselov, A.P. Finite-zone, two-dimensional, potential Schrödinger operators. Explicit formula and evolutions equations, *Sov. Math. Dokl.* **30** (1984), 588591.

- [GN] Grinevich, P. G. and Novikov, S. P. A two-dimensional inverse scattering problem for negative energies, and generalized-analytic functions. I. Energies lower than the ground state. (Russian) *Funktsional. Anal. i Prilozhen.* **22** (1988), no. 1, 233-3, 96; translation in *Funct. Anal. Appl.* **22** (1988), no. 1, 192-7
- [LMSS] Lassas, M., Mueller, J. L., Siltanen, S., Stahel, A. The Novikov-Veselov equation and the inverse scattering method, Part I: Analysis. *Phys. D* **241** (2012), no. 16, 1322-1335.
- [P] Perry, P. Miura maps and inverse scattering for the Novikov-Veselov equation. *Anal. PDE* **7** (2014), no. 2, 311-343.
- [CMMPSS] Croke, R., Mueller, J., Music, M., Perry, P., Siltanen, S., Stahel, A. The Novikov-Veselov equation: theory and computation. *Contemporary Mathematics* **635** (2015), 25-74
- [6] In case of the Laplace transformation the contour of integration must be chosen lying inside of a sector where $\cos(3\theta) > 0$
- [7] We would like to remind the reader that in literature the term *generalized Airy function* is commonly assigned to the solutions of the second order O.D.E. $w''(x) = x^n w(x)$; hence the addition of the term *higher-order* in our case is necessary to avoid a possible confusion.